Sequences of pseudo-Anosov mapping classes and their asymptotic behavior

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1 Introduction

Consider a surface $S_{g,n}$ with genus g and n punctures with negative Euler characteristic. The group of self homeomorphisms of the surface up to isotopy is called the *mapping class group*, $\mathrm{Mod}^+(S_{g,n})$. By the Nielsen-Thurston classification [bDKM11] an element of the mapping class group is either periodic, reducible(i.e. there exists a non-trivial invariant curve), or pseudo-Anosov. A pseudo-Anosov element is one which fixes a pair of transverse measured singular foliations,

$$\phi((\mathcal{F}^{\pm}, \mu^{\pm})) = (\mathcal{F}^{\pm}, \lambda^{\pm 1}\mu^{\pm})$$

up to scaling the measures $\mu^{\pm 1}$ by a constants $\lambda^{\pm 1}$ where $\lambda > 1$. The number λ is called the *dilatation* of ϕ. The set of dilatations is discrete and therefore bounded away from 1 [Yoc81] [Iva90]. Furthermore for fixed g and n the minimum, $\delta_{g,n}$, is achieved by some mapping class $\phi \in \mathrm{Mod}^+(S_{g,n})$. One open question about the spectrum of dilataitons is the following one.

Question 1.1. *What is the value of $\delta_{g,n}$ given (g, n) defining a surface of negative Euler characteristic?*

This question is only answered for a handful of cases of small g and n, see [KLS02], [HS07], [CH08]. More is known about the general behavior of these numbers. Penner explored the behavior for closed surfaces and proved the following theorems.

Theorem 1.2. *[Pen91]*

$$\log(\delta_{g,n}) \geq \frac{\log(2)}{12g - 12 + 4n} \tag{1}$$

Theorem 1.3. *[Pen91]*

$$\log(\delta_{g,0}) \asymp \frac{1}{g}. \tag{2}$$

Tsai [Tsa09](cf [HK06]) continued this investigation for punctured surfaces and showed that for $g = 0$ or $g = 1$ and n even the behavior is:

$$\log \delta_{g,n} \asymp \frac{1}{n}.$$

However for surfaces of genus $g > 1$ the minimal dilatations behave like,

$$\log(\delta_{g,n}) \asymp \frac{\log(n)}{n}.$$

This lead to the following question.

Question 1.4. *[Tsa09] What is the behavior for the minimal dilatations for different sequences of (g, n)?*

In this paper we provide a partial answer to Tsai's question.

Theorem 1.5. *Given any rational number r the asymptotic behavior along the ray defined by $g = rn$ is,*

$$\log(\delta_{g,n}) \asymp \frac{1}{|\chi(S_{g,n})|},$$

where $\chi(S_{g,n})$ is the Euler characteristic of the surface $S_{g,n}$.

The proof follows Penner's proof of Theorem 1.3. In [Pen91] Penner proves a general lower bound and defines a sequence of pseudo-Anosov mapping classes $\phi_g : S_{g,0} \to S_{g,0}$ such that $\lambda((\phi_g)^g)$ is bounded by some constant. We use Penner's lower bound [Pen91] and generalize his examples. This generalization allows us to construct sequences with dilatation bounded by some constant multiple of $\frac{1}{|\chi(S_{g,n})|}$. Given certain choices we can find bounding examples for the upper bound in Theorem 1.5.

In Section 2 we recall some known results about pseudo-Anosov mapping classes and train tracks and some techniques of Penner's used in providing the upper bound for closed surfaces. In Section 3 we define generalized Penner sequences and begin to prove that such a sequence ϕ_n has $\log(\lambda(\phi_n)) \asymp \frac{1}{|\chi(S_{g,n})|}$. In Section 4 we apply our construction and its behavior to the proof of Theorem 1.5.

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2 Background

In this section we recall some facts about pseudo-Anosov mapping classes and train tracks that will be used later in this paper.

Dehn showed in [Deh38] that the mapping class group is generated by finitely many Dehn twists d_x where x is a simple closed curve in the surface. The following theorem of Penner's gives a partial answer to the question of which words of Dehn twists define pseudo-Ansov mapping classes.

Theorem 2.1 (Penner's Semigroup Criterion)**.** *[Pen88] Suppose $\mathcal{C}$ and $\mathcal{D}$ are each disjointly embedded collections of simple closed curves in an oriented surface S. Suppose $\mathcal{C}$ interesects $\mathcal{D}$ minimally and $\mathcal{C} \cup \mathcal{D}$ fills S (i.e. the connected components of $S/C \cup D$ have non-negative Euler characteristic). Let $R(\mathcal{C}^+, \mathcal{D}^-)$ be the semigroup generated by $\{d_c \mid c \in \mathcal{C}\} \cup \{d_d^{-1} \mid d \in \mathcal{D}\}$. If $\omega \in R(\mathcal{C}^+, \mathcal{D}^-)$, then ω is pseudo-Anosov if each d_c and d_d^{-1} appear in ω.*

Every pseudo-Anosov mapping class $\phi : S_{g,n} \to S_{g,n}$ has an associated train track $\tau \subset S_{g,n}$ such that $\phi(\tau)$ is smoothly homotopic into τ, or $\phi(\tau)$ is *carried* by τ. This homotopy defines a transition matrix $T[a_{i,j}]$ on the edges of τ where the entry $a_{i,j}$ is the incidence of the edge i with the edge j after applying ϕ followed by the homotopy. This matrix defines a linear action on the vector space of admissable measures of τ. Each admissable measure of τ defines a measured singular foliation of $S_{g,n}$ and so finding the real eigenvectors of T is equivalent to finding the invariant foliations where the real eigenvalue is the dilatation. For a further discussion of train tracks see [PH92] [Pen88].

Recall that a non-negative matrix M such that M^n is positive for some $n > 0$ is said to be Perron-Frobenious. Such a matrix has a unique real eigenvector whose eigenvalue is the spectral radius of the matrix. We will use the following lemma to bound the spectral radii of these matrices. The result is well known and we include the proof for the convenience ofthe reader.

Lemma 2.2. *The spectral radius of a Perron-Frobenius matrix is bounded by the largest column sum.*

Proof: Let M be a Perron-Frobenius matrix with spectral radius λ and corresponding eigenvector v with norm 1. This eigenvector is real and positive.

$$\mid vM \mid = \lambda = \frac{\sum_{i=1}^{n} v_i m_{ij}}{v_j} \text{ for all } j = 1...n.$$

Let v_j be the largest component of v. Then we have:

$$\mid vM \mid = \lambda = \sum_{i=1}^{n} \frac{v_i}{v_j} m_{ij}.$$

But each term $\frac{v_i}{v_j} \leq 1$ and each term $m_{ij} \geq 0$ and so we have the inequality:

$$\lambda \leq \sum_{i=1}^{n} m_{ij}.$$

Thus λ is bounded by the largest column sum.$\square$

We will construct pseudo-Anosov maps and corresponding Perron-Frobenius matrices.

As we have already mentioned a pseudo-Anosov mapping class defines a pair of transverse measured singular foliations $(\mathcal{F}^{\pm}, \mu^{\pm})$. The following lemma tells us when a pseudo-Anosov mapping class extends under the forgetful map to another with the same dilatation, see [HK06].

Lemma 2.3. *If ϕ is a pseudo-Anosov mapping class on the surface $S_{g,n}$, some subset of the punctures I is fixed setwise, and if none of the points in I are 1-pronged then the punctures may be filled in and the induced mapping class $\tilde{\phi}$ is pseudo-Anosov with $\lambda(\phi) = \lambda(\tilde{\phi})$.*

An *m-gon* is a possibly punctured disc in $S_{g,n} \backslash \tau$ with m cusps. We will be able to apply Lemma 2.3 with the use of the following Lemma, see [PH92].

Lemma 2.4. *Let ϕ be a pseudo-Anosov mapping class and τ a compatible train track. Then we have the following:*

1. *The singularities and punctures of S defined by the stable foliation of ϕ are in one-to-one correspondence with the the m-gons of τ, and*

2. *a singularity or puncture is m-pronged if and only if it is contained in an m-gon of τ.*

We will use this information in order to construct examples, find transition matrices that bound their dilatation and lastly extend these examples to ones suitable for the gn-rays we are interested in.

3 Penner Sequences

In this section we will define Penner sequences and give the stepping stones to prove the following theorem.

Theorem 3.1. *Given a Penner sequence of mapping classes $\phi_m : F_m \to F_m$ there is a constant P such that*

$$\log(\lambda(\phi_m)) \asymp \frac{P}{\mid \chi(F_m) \mid}$$

where $\chi(F_m)$ is the Euler characteristic of the surface F_m.

First we contstruct the surfaces the mapping classes are defined on. Consider an oriented surface with boundary and punctures, $S_{g,n,b}$, with two sets of disjoint arcs on the boundary components a^- and a^+ such that

$$a^- \cap a^+ = \emptyset$$

and an orientation reversing homeomorphism,

$$\iota : a^+ \to a^-.$$

Let Σ_i be homeomorphic copies of $S_{g,n,b}$ and let $h_i : S_{g,n,b} \to \Sigma_i$ be a homeomorphism for each $i \in \mathbb{Z}$. Set

$$F_\infty = \bigcup_{i \in \mathbb{Z}} \Sigma_i / \sim,$$

where $y_i \sim y_j$ if, for some $x \in a^+$ and $k \in \mathbb{Z}$,

$$(y_i, y_j) = (h_k(x), h_{k+1}(\iota(x))).$$

The action

$$\rho = h_{i+1} h_i^{-1}$$

acts properly discontinuously on F_∞. Then we define

$$F_m = F_\infty / \rho^m.$$

We then pick two sets of multicurves C and D on Σ_1 satisfying Theorem 2.1. A *connecting curve* is a curve, γ, on F_∞ such that $\gamma \subset \Sigma_1 \cup \Sigma_2$, $C \cup \rho(C) \cup \gamma$ is a multicurve, γ intersects $D \cup \rho(D)$ minimally, and the set of curves,

$$J = \{\rho^i(C \cup D \cup \gamma)\}_{-\infty}^{\infty}$$

fills the surface F_∞.

Definition 3.2. A sequence of mapping classes $\phi_m : F_m \to F_m$ is called a Penner sequence if for some (C, D) as in Theorem 2.1,

$$\phi_m = \rho d_\gamma \omega,$$

where $\omega \in R(C^+, D^-)$ is pseudo-Anosov on Σ_1.

Next we want to show that these mapping classes are pseudo-Anosov. We start with the following lemma about train tracks. Here we allow our train tracks to have bigons.

Lemma 3.3. *Given a Penner sequence ϕ_m there exists invariant train tracks on each surface F_m such that:*

1. *The curves in C and D and the connecting curve are carried on the train track.*
2. *The images of the curves used to define ω and the image of the connecting curve under the map ϕ_m are carried on the train track.*

Proof: We construct a train track on the surface F_∞ and then project it to F_m. Consider the set of curves J in the definition of F_∞. Then assign positive orientation to all curves,

$$\{\rho^i(C \cup \gamma)\}_{-\infty}^{\infty},$$

and negative orientation to all curves,

$$\{\rho^i(D)\}_{-\infty}^{\infty}.$$

We then smooth the intersections according to Figure 3.

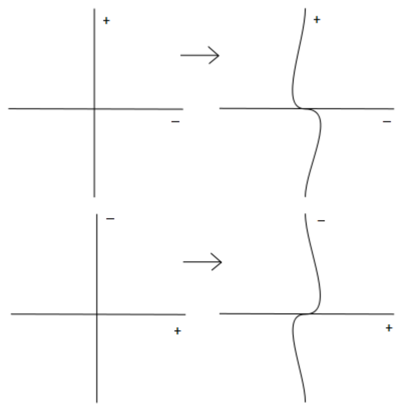

Figure 1: Smoothing

This provides a train track, τ, which may be projected to a track τ_m on F_m. It is easy to see that all the curves in J are carried by the train track. If we consider a curve x that is carried by τ_m and perform a Dehn twist by an element $y \in J$ then the resulting curve $d_y(x)$ is carried by the edges carrying x and the edges that carry the element y. Since the train track is symmetric with respect to the rotation map we are done.□

The next lemma allows us to compute the entries of a transition matrix on this train track.

Lemma 3.4. *Consider $x, y \in J$ where,*

$$J = \cup_{i=1\ldots m}\rho^i(C \cup D \cup \gamma).$$

If μ_x and μ_y are the elements of W_{τ_m} induced by the curves x and y and $d_x^\star$ is the map on W_{τ_m} induced by d_x then

$$d_x^\star(\mu_y) = \mu_y + i(x, y)\mu_x$$

Proof: Collapsing a curve is invariant up to homotopy so the edges that elements of J collapse to on the train track are well defined. Therefore we can consider a curve as a free homotopy class. If two curves $x, y \in J$ are in minimal position then performing a Dehn twist on y about x will send y to $d_x(y) = y + i(x, y)x$. We collapse the homology elements to τ_m and obtain the desired result.□

Lemma 3.5. *The matrix T_{ϕ_m} is Perron-Frobenius for each $m \geq 2$.*

Proof: This follows from the previous lemma and the fact that given the mapping class ϕ_m^m where we perform Dehn twists about all the curves in,

$$J_m = \cup_{i=1...m}\rho^i(C \cup D \cup \gamma),$$

each curve is connected to another through at most $m(r+s+1)$ curves. Therefore $T_{\phi_m}^{m^2(r+s+1)}$ is strictly positive where $r = \sharp C$ and $s = \sharp D$.$\square$

Remark: An admissable measure on a bigon track defines a measured foliation up to an equivalence of the admissable measures [PH92]. We only consider a subset of the admissable measures when computing transition matrices. Since we have show that the transition matrix on these measures is Perron-Frobenius the eigenvector is positive, defining an invariant foliation for the mapping class. We will prove that these mapping classes are pseudo-Anosov in the next section, therefore the invariant expanding foliation is unique and we need not worry about the equivalent measures or measures outside the considered subset.

4 Asymptotic Behavior

Proof of Theorem 3.1: The mapping classes $(\phi_m)^m$ are pseudo-Anosov by Penner's semigroup criteria and so the mapping classes ϕ_m are as well. Lemma 3.3 gives an invariant bigon train track.

As stated in Theorem 2.2 the spectral radius of a Perron-Frobenius matrix is bounded by the largest column sum of the matrix. So now we would like to compute the matrix defining the action on the transverse measures. This matrix will be Perron-Frobenius by Lemma 3.5 and can be computed using Lemma 3.4.

The map ρ permutes the curves of $J_m = \cup_{i=1...m}\rho^i(C \cup D \cup \gamma)$ and so the induced map on the space of weights spaned by $\mu_1...\mu_{m(r+s+1)}$, where $r = \sharp C$ and $s = \sharp D$, is defined by a block permutation matrix.

$$M_\rho = \begin{pmatrix} 0 & I & 0 & . & 0 \\ 0 & 0 & I & . & 0 \\ . & . & . & . & . \\ I & 0 & 0 & . & 0 \end{pmatrix}$$

The Dehn twist about the connecting curve gives the map with transition matrix defined by,

$$M_{d_\gamma} = \begin{pmatrix} U & 0 & 0 & . & 0 \\ V & I & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & I \end{pmatrix}.$$

Last the transition matrix for the map induced by the word ω is given below.

$$M_\omega = \begin{pmatrix} W & 0 & 0 & . & 0 \\ 0 & I & 0 & . & 0 \\ 0 & 0 & I & . & 0 \\ . & . & . & . & . \\ X & 0 & 0 & . & I \end{pmatrix}$$

Then the matrix for the map ϕ_m is given below by matrix multiplication after making the identifications $WU = Y$ and $XK = Z$.

$$M_{\phi_m} = \begin{pmatrix} 0 & Y & 0 & 0 & . & . & 0 \\ 0 & V & I & 0 & . & . & 0 \\ 0 & 0 & 0 & I & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & I \\ I & Z & 0 & 0 & . & . & 0 \end{pmatrix}$$

The matrix M is an $m \times m$ block matrix of $r+s+1 \times r+s+1$ blocks. The matrix Y depends on the word ω. The matrix V may have non-zero entries in the last column except that the last row must be zero since $\gamma \cap \rho(\gamma) = \emptyset$. The observation $V^2 = 0$ will be important later.

$$V = \begin{pmatrix} 0 & . & . & . & 0 & v_1 \\ 0 & . & . & . & 0 & v_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & v_{rs} \\ 0 & . & . & . & 0 & 0 \end{pmatrix}$$

The matrix Z will depend on ω as well but only has non-zero entries in the last row. Now we want to consider the matrix M^m. Inductively we see that for $1 < k < m$ the matrix M^k is given by the following matrix. Here we use the fact that V^2 is the zero matrix.

$$M^k_{\phi_m} = \begin{pmatrix} 0 & 0 & 0 & . & YV & Y & 0 & 0 & . & 0 \\ . & . & . & . & 0 & V & I & 0 & . & . \\ . & . & . & . & . & 0 & 0 & I & . & 0 \\ 0 & 0 & . & . & . & . & . & 0 & . & I \\ I & Z & 0 & . & . & . & . & . & . & 0 \\ 0 & Y+ZV & Z & . & . & . & . & . & . & . \\ . & YV & Y+ZV & . & . & . & . & . & . & . \\ . & 0 & YV & . & . & . & . & . & . & . \\ . & . & 0 & . & 0 & . & . & . & . & . \\ . & . & . & . & C & 0 & . & . & . & . \\ 0 & 0 & 0 & . & Y+ZV & Z & 0 & 0 & . & 0 \end{pmatrix}$$

Then we can find the transition matrix for the mth iterate.

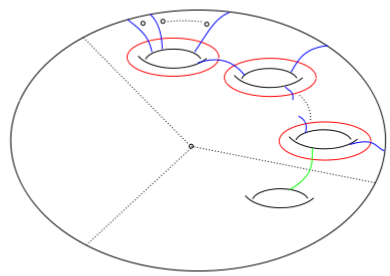

Figure 2: Choice for curves

$$M^m = \begin{pmatrix} Y & YZ & 0 & . & . & 0 & YV \\ V & Y+ZV & Z & . & . & . & 0 \\ 0 & YV & Y+ZV & . & . & . & . \\ . & 0 & YV & . & . & . & . \\ . & . & 0 & . & . & 0 & . \\ . & . & . & . & . & Z & 0 \\ 0 & . & . & . & . & Y+ZV & Z \\ C & 0 & 0 & . & . & YV & Y+ZV \end{pmatrix}$$

Then this matrix is Perron-Frobenius by Lemma 3.5 and by Theorem 2.2 the spectral radius is bounded by the largest column sum. A block column sum is either equal to a column sum of $YV+Z+ZV$, $YZ+Y+ZV+YV$, or $Y+V+Z$. Therefore the dilatation of the mth iterate is bounded by a constant, say P. This tells us that

$$\log(\lambda(\phi_m)) \leq \frac{P}{m}.$$

Then Theorem 1.2 with the upper bound just given finishes the proof.□

Next we use this to prove Theorem 1.5

Proof of Theorem 1.5: With Penner's lower bound we only need the upper bound to prove the asymptotic behavior. Suppose a ray has slope $\frac{p}{q}$ with $(p,q) = 1$ then let $S_{g,n,1}$ have $g = p$, and $n = q$ and create any Penner sequence with a^+ being an arc on the boundary component and a^- another disjoint arc on the same boundary component. This sequence of mapping classes then has the required upper bound on dilatation. This gives sequences on gn-rays through $(2,0)$. Further if we choose our curves as in Figure 4, which is shown with a chosen connecting curve as well, then we can find a train track for ϕ_m given in Figure 4.

From this we can see by Theorem 2.4 that the two fixed punctures are not 1-pronged. Filling in both fixed punctures we obtain sequences of mapping classes with the same dilatation and two fewer punctures, the sequences for the lines passing through the origin.□

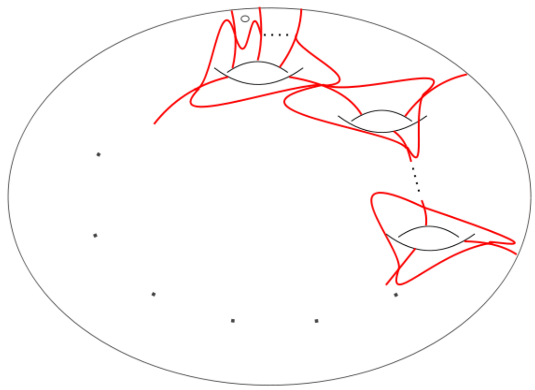

Figure 3: Train track

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